Complex analysis 2

Frames 1 to 79

Learning outcomes

When you have completed this Programme you will be able to:

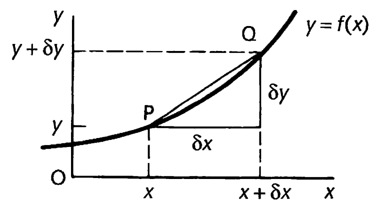
- Appreciate when the derivative of a function of a complex variable exists
- Understand the notions of regular functions and singularities and be able to obtain the derivative of a regular function from first principles
- Derive the Cauchy–Riemann equations and apply them to find the derivative of a regular function
- Understand the notion of an harmonic function and derive a conjugate function
- Evaluate line and contour integrals in the complex plane
- Derive and apply Cauchy's theorem
- Apply Cauchy's theorem to contours around regions that contain singularities
- Define the essential characteristics of and conditions for a conformal mapping
- Locate critical points of a function of a complex variable
- Determine the image in the w-plane of a figure in the z-plane under a conformal transformation $w = f(z)$
- Describe and apply the Schwarz–Christoffel transformation

1

In the previous Programme we introduced the ideas of mapping from one complex plane to another and considered some of the more common transformation functions. Now we pursue our consideration of the complex variable a little further.

Differentiation of a complex function

In differentiation of a function of a single real variable, $y = f(x)$, the derivative of y with respect to x can be defined as the limiting value of $\dfrac{(y + \delta y) - y}{\delta x}$ as δx tends to zero.



$$y = f(x) \quad \delta y = f(x + \delta x) - f(x)$$

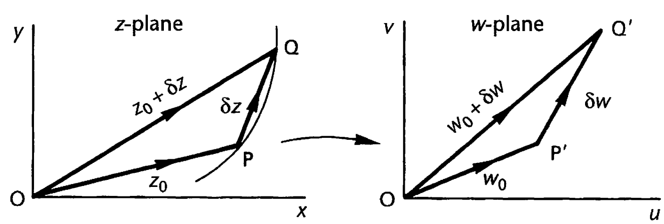
$$\text{i.e. } \frac{\mathrm{d}y}{\mathrm{d}x} = \underset{\delta x \to 0}{Lim}\left\{\frac{f(x + \delta x) - f(x)}{\delta x}\right\}$$

In considering the differentiation of a function of a complex variable, $w = f(z)$, the derivative of w with respect to z can similarly be defined as the limiting value of .... as δz tends to zero.

2

$$\boxed{\frac{(w + \delta w) - w}{\delta z} \quad \text{i.e.} \quad \frac{f(z + \delta z) - f(z)}{\delta z}}$$

Now, of course, we are dealing in vectors.



If P and Q in the z-plane map onto P′ and Q′ in the w-plane, then

$$\mathrm{P'Q'} = \delta w = (w_0 + \delta w) - w_0 = f(z_0 + \delta z) - f(z_0)$$

Therefore, the derivative of w at P′ $(z = z_0)$ is the limiting value of $\dfrac{\delta w}{\delta z}$ as

$$\delta z \to 0, \text{ i.e. } \left[\frac{\mathrm{d}w}{\mathrm{d}z}\right]_{z_0} = \underset{\delta z \to 0}{Lim}\left\{\frac{f(z_0 + \delta z) - f(z_0)}{\delta z}\right\} = \underset{\mathrm{Q} \to \mathrm{P}}{Lim}\left(\frac{\mathrm{P'Q'}}{\mathrm{PQ}}\right)$$

If this limiting value exists – which is not always the case as we shall see – the function $f(z)$ is said to be *differentiable at* P.

Also, if $w = f(z)$ and $f'(z)$ has a limit for all points z_0 within a given region for which $w = f(z)$ is defined, then $f(z)$ is said to be differentiable in that region. From this, it follows that the limit exists whatever the path of approach from Q $(z = z_0 + \delta z)$ to P $(z = z_0)$.

Regular function

A function $w = f(z)$ is said to be *regular* (or analytic) at a point $z = z_0$, if it is defined and single-valued, and has a derivative at every point at and around z_0. Points in a region where $f(z)$ ceases to be regular are called *singular points*, or *singularities*.

A function of a complex variable that is analytic over the entire finite complex plane is called an *entire* function. Examples of entire functions are polynomials, e^z, $\sin z$ and $\cos z$.

We have introduced quite a few new definitions, so let us pause here while you make a note of them. We shall be meeting the various terms quite often.

3

In those cases where a derivative exists, the usual rules of differentiation apply. For example, the derivative of $w = z^2$ can be found from first principles in the normal way.

$$w = z^2 \quad \therefore \quad w + \delta w = (z + \delta z)^2 = z^2 + 2z\delta z + \delta z^2$$

$$\therefore \quad \delta w = 2z\delta z + \delta z^2 \quad \therefore \quad \frac{\delta w}{\delta z} = 2z + \delta z$$

$\therefore \quad \dfrac{\mathrm{d}w}{\mathrm{d}z} = \underset{\delta z \to 0}{Lim}\,(2z + \delta z) = 2z$ and does not depend on the path along which δz tends to zero.

That was elementary. Here is a rather different one.

Example

To find the derivative of $w = z\bar{z}$ where $z = x + jy$ and $\bar{z} = x - jy$.

We have $w = z\bar{z} \quad \therefore \quad w + \delta w = (z + \delta z)(\bar{z} + \delta\bar{z})$ from which

$$\frac{\delta w}{\delta z} = \ldots\ldots\ldots\ldots$$

4

$$\boxed{\frac{\delta w}{\delta z} = \bar{z} + z\frac{\delta\bar{z}}{\delta z} + \delta\bar{z}}$$

Because

$$w + \delta w = (z + \delta z)(\bar{z} + \delta\bar{z}) = z\bar{z} + \bar{z}\delta z + z\delta\bar{z} + \delta z\delta\bar{z}$$

$$\therefore \quad \delta w = \bar{z}\delta z + z\delta\bar{z} + \delta z\delta\bar{z} \quad \therefore \quad \frac{\delta w}{\delta z} = \bar{z} + z\frac{\delta\bar{z}}{\delta z} + \delta\bar{z}$$

Now since $z = x + jy$ and $\bar{z} = x - jy$, we can express $\dfrac{\delta w}{\delta z}$ in terms of x and y. $\quad \dfrac{\delta w}{\delta z} = \ldots\ldots\ldots\ldots$

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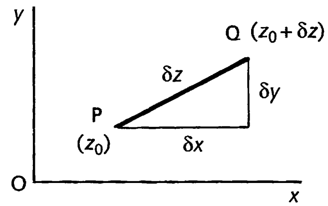
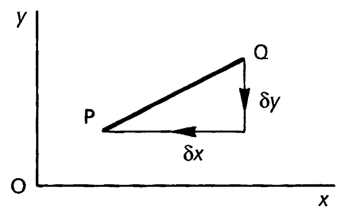
$$\boxed{\frac{\delta w}{\delta z} = (x - jy) + (x + jy)\left\{\frac{\delta x - j\delta y}{\delta x + j\delta y}\right\} + \delta x - j\delta y}$$

Because

$$\left.\begin{aligned} z &= x + jy \quad \therefore\ \delta z = \delta x + j\delta y \\ \overline{z} &= x - jy \quad \therefore\ \delta\overline{z} = \delta x - j\delta y \end{aligned}\right\} \quad \therefore\ \frac{\delta\overline{z}}{\delta z} = \frac{\delta x - j\delta y}{\delta x + j\delta y}$$

Then $\dfrac{\delta w}{\delta z} = \overline{z} + z\dfrac{\delta\overline{z}}{\delta z} + \delta\overline{z}$ gives the expression quoted above.





The next step is to reduce δz to zero. But δz consists of $\delta x + j\delta y$ and so reducing δz to zero can be done in one of two ways.

(1) First let $\delta y \to 0$ and afterwards let $\delta x \to 0$.

If $\delta y \to 0$, $\quad \dfrac{\delta w}{\delta z} = x - jy + (x + jy)\dfrac{\delta\overline{x}}{\delta x} + \delta\overline{x}$

Then $\dfrac{dw}{dz} = \underset{\delta x \to 0}{Lim}\{x - jy + x + jy + \delta x\}$

$= \dots\dots\dots\dots$

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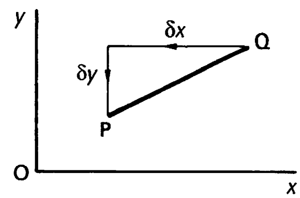
$$\boxed{\frac{dw}{dz} = 2x}$$

On the other hand, we could have reduced δz to zero in the second way.



(2) First let $\delta x \to 0$ and afterwards let $\delta y \to 0$.

We have $\quad \dfrac{\delta w}{\delta z} = x - jy + (x + jy)\left\{\dfrac{\delta x - j\delta y}{\delta x + j\delta y}\right\} + \delta x - j\delta y$

If $\delta x \to 0 \quad \dfrac{\delta w}{\delta z} = x - jy + (x + jy)(-1) - j\delta y = -j2y - j\delta y$

Then $\quad \dfrac{dw}{dz} = \lim\limits_{\delta y \to 0} \{-j2j - j\delta y\} = -j2y$

So, in the first case, $\dfrac{dw}{dz} = 2x$ and in the second case $\dfrac{dw}{dz} = -j2y$.

These two results are clearly not the same for all values of x and y – with one exception, i.e.

....

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$$\boxed{\text{when } x = y = 0}$$

Therefore $w = z\bar{z}$ is a function that has no specific derivative, except at $z = 0$ – and there are others. It would be convenient, therefore, to have some form of test to see whether a particular function $w = f(z)$ has a derivative $f'(z)$ at $z = z_0$. This useful tool is provided by the Cauchy–Riemann equations.

Cauchy–Riemann equations

The development is very much along the same lines as in the previous example. If $w = f(z) = u + jv$, we have to establish conditions for $w = f(z)$ to have a derivative at a given point $z = z_0$.

$$w = u + jv \quad \therefore \ \delta w = \delta u + j\delta v; \qquad z = x + jy \quad \therefore \ \delta z = \delta x + j\delta y$$

$$\text{Then} \quad f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0}\left\{\frac{\delta u + j\delta v}{\delta z}\right\} = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}}\left\{\frac{\delta u + j\delta v}{\delta x + j\delta y}\right\} \qquad (1)$$

(a) Let $\delta x \to 0$, followed by $\delta y \to 0$

Then from (1) above, $\quad f'(z) = \dfrac{dw}{dz} = \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{dw}{dz} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}}$$

Because

$$f'(z) = \underset{\delta y \to 0}{Lim}\left\{\frac{\delta u + j\delta v}{j\delta y}\right\} = \underset{\delta y \to 0}{Lim}\left\{\frac{\delta v}{\delta y} - j\frac{\delta u}{\delta y}\right\} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y} \qquad (2)$$

We use the 'partial' notation since u and v are functions of both x and y.

Or (b) Let $\delta y \to 0$, followed by $\delta x \to 0$.

This gives ....

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$$\boxed{\frac{dw}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}}$$

Because

$$f'(z) = \underset{\delta x \to 0}{Lim}\left\{\frac{\delta u + j\delta v}{\delta x}\right\} = \underset{\delta x \to 0}{Lim}\left\{\frac{\delta u}{\delta x} + j\frac{\delta v}{\delta x}\right\} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} \qquad (3)$$

If the results (2) and (3) are to have the same value for $f'(z)$ irrespective of the path chosen for δz to tend to zero, then

....

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$$\boxed{\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}}$$

Equating real and imaginary parts, this gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are the *Cauchy–Riemann equations*.

So, to sum up:

A necessary condition for $w = f(z) = u + jv$ to be regular at $z = z_0$ is that u, v and their partial derivatives are continuous and that in the neighbourhood of $z = z_0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Make a note of this important result – then move on to the next frame

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We said earlier that where a function fails to be regular, a *singular point*, or *singularity* occurs, for example where $w = f(z)$ is not continuous or where the Cauchy–Riemann test fails.

Exercise

Determine where each of the following functions fails to be regular, i.e. where singularities occur.

1 $w = z^2 - 4$

2 $w = \dfrac{z}{z-2}$

3 $w = \dfrac{z+5}{z+1}$

4 $w = \dfrac{1}{(z-2)(z-3)}$

5 $w = z\bar{z}$

6 $w = \dfrac{x+jy}{x^2+y^2}$

Finish all six: then check with the next frame

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Conclusions:

1 Putting $z = x + jy$, the Cauchy–Riemann conditions are satisfied everywhere. Therefore, no singularity in $w = z^2 - 4$.

2 The function becomes discontinuous at $z = 2$. Singularity at $z = 2$.

3 The function is discontinuous at $z = -1$. Singularity at $z = -1$.

4 Singularities at $z = 2$ and $z = 3$.

5 We have already seen that $w = z\bar{z}$ has no derivative for all values of z apart from $z = 0$. All points on $w = z\bar{z}$ are singularities.

6 Singularity occurs where $x^2 + y^2 = 0$, i.e. $x = 0$ and $y = 0$ $\therefore$ $z = 0$. At all other points the Cauchy–Riemann equations do not hold.

Harmonic functions

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If a function of two real variables $f(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0$$

then we say that $f(x, y)$ is an *harmonic* function. It is relatively straightforward to demonstrate that the real and imaginary parts of an analytic function are both harmonic.

▶

Let $f(z) = u(x, y) + jv(x, y)$ be an analytic function in some region of the z-plane. Because $f(z)$ is analytic the Cauchy–Riemann equations hold true. That is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiating the first with respect to x and the second with respect to y shows us that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2}$$

$$\text{and so} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

By a similar reasoning

$$\frac{\partial^2 \ldots}{\partial x^2} + \frac{\partial^2 \ldots}{\partial y^2} = 0$$

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$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

Because

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\text{and so} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

The functions $u(x, y)$ and $v(x, y)$ are called *conjugate* functions. In addition, the curves $u = \text{constant}$, $v = \text{constant}$ are orthogonal.

Example 1

Show that the real and imaginary parts of the function defined by $f(z) = z^2$ are harmonic.

$$\begin{aligned} f(z) &= z^2 \\ &= (x + jy)^2 \\ &= (x^2 - y^2) + 2jxy \end{aligned}$$

and so $u = x^2 - y^2$ and $v = 2xy$ and therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \ldots\ldots\ldots\ldots \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ and } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0}$$

Because

$$\frac{\partial u}{\partial x}=2x \text{ so } \frac{\partial^2 u}{\partial x^2}=2 \text{ and } \frac{\partial u}{\partial y}=-2y \text{ so } \frac{\partial^2 u}{\partial y^2}=-2$$

$$\text{therefore } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

and

$$\frac{\partial v}{\partial x}=2y \text{ so } \frac{\partial^2 v}{\partial x^2}=0 \text{ and } \frac{\partial v}{\partial y}=2x \text{ so } \frac{\partial^2 v}{\partial y^2}=0$$

$$\text{therefore } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Example 2

Show that $u(x, y)=x^3y-y^3x$ is an harmonic function and find the function $v(x, y)$ that ensures that $f(z)=u(x,u)+jv(x, y)$ is analytic. That is, find the function $v(x, y)$ that is conjugate to $u(x, y)$.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\dots\dots\dots\dots$$

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$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$

Because

$$\frac{\partial u}{\partial x}=3x^2y-y^3 \text{ so } \frac{\partial^2 u}{\partial x^2}=6xy \text{ and } \frac{\partial u}{\partial y}=x^3-3y^2x \text{ so } \frac{\partial^2 u}{\partial y^2}=-6xy$$

$$\text{therefore } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

This means that $u(x, y)=x^3y-y^3x$ is harmonic.

Now, if $f(z)=u(x,u)+jv(x, y)$ is analytic then $u(x, y)$ and $v(x, y)$ satisfy the .... equations.

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$$\boxed{\text{Cauchy–Riemann}}$$

That is

$$\frac{\partial u}{\partial x} = 3x^2y - y^3 = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = x^3 - 3y^2x = -\frac{\partial v}{\partial x}$$

Integrating $\frac{\partial v}{\partial y} = 3x^2y - y^3$ with respect to y gives

$$v(x, y) = \ldots\ldots\ldots\ldots$$

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$$\boxed{v(x, y) = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + a(x)}$$

Because

$\frac{\partial v}{\partial y} = 3x^2y - y^3$ and so x is treated as a constant and the integral of y^n is $y^{n+1}/(n+1)$.

Did you miss the constant term in the form of $a(x)$? *Because x is treated as a constant, the integration determines y up to an expression involving x.* Differentiate the result with respect to y and you will reclaim the original form for $\frac{\partial v}{\partial y}$.

Now, differentiating this expression with respect to x gives

$$\frac{\partial v}{\partial x} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\partial v}{\partial x} = 3xy^2 + a'(x)}$$

Because

$v(x, y) = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + a(x)$ and so $\frac{\partial v}{\partial x} = 3xy^2 + a'(x)$ and this is equal to $-\frac{\partial u}{\partial y}$. Now $-\frac{\partial u}{\partial y} = -x^3 + 3y^2x$ and so

$$a'(x) = \ldots\ldots\ldots\ldots \quad \text{giving} \quad a(x) = \ldots\ldots\ldots\ldots$$

$$\text{Therefore} \quad v(x, y) = \ldots\ldots\ldots\ldots$$

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$$a'(x) = -x^3 \text{ giving } a(x) = -\frac{x^4}{4} + C.$$
$$\text{Therefore } v(x, y) = \frac{3x^2y^2}{2} - \frac{y^4}{4} - \frac{x^4}{4} + C$$

Because

Comparing $\frac{\partial v}{\partial x} = 3xy^2 + a'(x)$ and $-\frac{\partial u}{\partial y} = -x^3 + 3y^2x$

where $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ then it is seen that $a'(x) = -x^3$.

Therefore $a(x) = -\frac{x^4}{4} + C$ giving $v(x, y) = \frac{3x^2y^2}{2} - \frac{y^4}{4} - \frac{x^4}{4} + C$

Try one for yourself.

Example 3

Given $u(x, y) = e^{-x}\cos y$, show that $u(x, y)$ is an harmonic function and find the function $v(x, y)$ that ensures that $f(z) = u(x, y) + jv(x, y)$ is analytic. That is, find the function $v(x, y)$ that is conjugate to $u(x, y)$.

$$\frac{\partial^2 \ldots}{\partial x^2} + \frac{\partial^2 \ldots}{\partial y^2} = \ldots\ldots\ldots\ldots$$

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Because

$u = e^{-x}\cos y$ so $\frac{\partial u}{\partial x} = -e^{-x}\cos y$ and $\frac{\partial^2 u}{\partial x^2} = e^{-x}\cos y$.

Also $\frac{\partial u}{\partial y} = -e^{-x}\sin y$ so $\frac{\partial^2 u}{\partial y^2} = -e^{-x}\cos y$. Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, that is $u(x, y)$ is harmonic. The conjugate function $v(x, y)$ is then

$$v(x, y) = \ldots\ldots\ldots\ldots$$

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$$v = -e^{-x}\sin y + C$$

Because

By the Cauchy–Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -e^{-x}\cos y$. Integrating with respect to y gives $v = -e^{-x}\sin y + a(x)$. Differentiating this with respect to x gives $\frac{\partial v}{\partial x} = e^{-x}\sin y + a'(x)$.

Now, by the other Cauchy–Riemann equation $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\sin y$, so that $a'(x) = 0$ giving $a(x) = C$. Therefore, $v = -e^{-x}\sin y + C$.

Now we shall look at complex integration. Move to the next frame

Complex integration

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At the beginning of this Programme, we defined differentiation with respect to z in the case of a complex function, since z is a function of two independent variables x and y, i.e. $z = x + jy$. Complex integration is approached in the same way.

$z = x + jy$ and $w = f(z) = u + jv$ where u and v are also functions of x and y.

Also $\mathrm{d}z = \mathrm{d}x + j\,\mathrm{d}y$ and $\mathrm{d}w = \mathrm{d}u + j\,\mathrm{d}v$

$$\therefore \quad \int w\,\mathrm{d}z = \int f(z)\,\mathrm{d}z = \int (u + jv)(\mathrm{d}x + j\,\mathrm{d}y)$$

$$= \int \{(u\,\mathrm{d}x - v\,\mathrm{d}y) + j(v\,\mathrm{d}x + u\,\mathrm{d}y)\}$$

$$\therefore \quad \int f(z)\,\mathrm{d}z = \int (u\,\mathrm{d}x - v\,\mathrm{d}y) + j\int (v\,\mathrm{d}x + u\,\mathrm{d}y)$$

That is, the integral reduces to two real-variable integrals

$$\int (u\,\mathrm{d}x - v\,\mathrm{d}y) \quad \text{and} \quad \int (v\,\mathrm{d}x + u\,\mathrm{d}y)$$

Note that each of these two integrals is of the general form $\int (P\,\mathrm{d}x + Q\,\mathrm{d}y)$ which we met before during our work on *line integrals* and, in the complex plane, this rather neatly leads us into *contour integration*.

Let us make a fresh start

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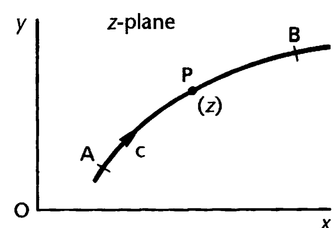
Contour integration – line integrals in the *z*-plane

If z moves along the curve c in the z-plane and at each position z has associated with it a function of z, i.e. $f(z)$, then summing up $f(z)$ for all such points between A and B means that we are evaluating a line integral in the z-plane between A $(z = z_1)$ and B $(z = z_2)$ along the curve c,



i.e. we are evaluating $\int_c f(z)\,\mathrm{d}z$ where c is the particular path joining A to B.

The evaluation of line integrals in the complex plane is known as *contour integration*. Let us see how it works in practice.

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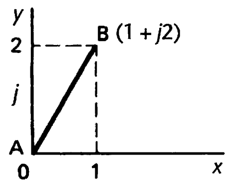
Example

Evaluate the integral $\int_c f(z)\,dz$ where $f(z) = (z - j)^2$ and c is the straight line joining A ($z = 0$) to B ($z = 1 + j2$).



$$z = x + jy; \quad dz = dx + j\,dy$$

$$f(z) = (z - j)^2 = \{x + j(y - 1)\}^2 = x^2 - (y - 1)^2 + j\,2x(y - 1)$$

$$\therefore\ I = \int \{(x^2 - y^2 + 2y - 1) + j(2xy - 2x)\}\{dx + j\,dy\}$$

$$= \int \{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dy\}$$

$$+ j\int \{(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)\,dy\}$$

Now the equation of AB is $y = 2x$. $\therefore$ $dy = 2\,dx$ and substituting these in the expression for I, between the limits $x = 0$ and $x = 1$, gives

$I = \dots\dots\dots\dots$ Finish it.

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$$\boxed{I = \frac{1}{3}(-2 + j)}$$

Because

$$I = \int_0^1 \{(x^2 - 4x^2 + 4x - 1)\,dx - (4x^2 - 2x)2\,dx\}$$

$$+ j\int_0^1 \{(4x^2 - 2x)\,dx + (2x^2 - 8x^2 + 8x - 2)\,dx\}$$

$$= \int_0^1 (-11x^2 + 8x - 1)\,dx + j\int_0^1 (-2x^2 + 6x - 2)\,dx$$

and this, by elementary integration, gives $I = \frac{1}{3}(-2 + j)$.

Now you will remember that, in general, the value of a line integral depends on the path of integration between the end points, but that the line integral $\int (P\,dx + Q\,dy)$ is independent of the path of integration in a simply connected region if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout the region.

In our example

$$I = \int \{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dy\}$$

$$+ j\int \{(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)\,dy\} \equiv I_1 + jI_2$$

If we apply the test to I_1, we get $\dots\dots\dots\dots$

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$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Because

$$\text{for } I_1 = \int\{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dx\} \equiv \int(P\,dx + Q\,dy)$$

$$\left.\begin{aligned} \therefore\ P &= x^2 - y^2 + 2y - 1 \quad \therefore\ \frac{\partial P}{\partial y} = -2y + 2 \\ Q &= -2xy + 2x \qquad\quad\ \ \therefore\ \frac{\partial Q}{\partial x} = -2y + 2 \end{aligned}\right\} \therefore\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

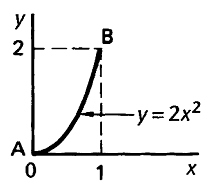
Similarly

$$\text{for } I_2 = \int\{(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)dy\} \equiv \int(Pdx + Qdy)$$

$$\left.\begin{aligned} \therefore\ P &= 2xy - 2x \qquad\qquad\ \therefore\ \frac{\partial P}{\partial y} = 2x \\ Q &= x^2 - y^2 + 2y - 1\ \therefore\ \frac{\partial Q}{\partial x} = 2x \end{aligned}\right\} \therefore\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Therefore, in this example, the value of the line integral is independent of the path of integration.

Just to satisfy our conscience, determine the value of the line integral between the same two end points, but along the parabola $y = 2x^2$.



$$f(z) = (z - j)^2$$

$$y = 2x^2 \quad \therefore\ dy = 4x\,dx$$

As before we have

$$I = \int\{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dy\}$$

$$+ j\int(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)\,dy\}$$

Substituting $y = 2x^2$ and $dy = 4x\,dx$, the evaluation gives

$$I = \dots\dots\dots\dots$$

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$$\boxed{I = \frac{1}{3}(-2 + j)}$$

We have

$$I = \int_0^1 \{(x^2 - 4x^4 + 4x^2 - 1)\,dx - (4x^3 - 2x)4x\,dx\}$$
$$+ j\int_0^1 \{(4x^3 - 2x)\,dx + (x^2 - 4x^4 + 4x^2 - 1)4x\,dx\}$$
$$= \int_0^1 (-20x^4 + 13x^2 - 1)\,dx + j\int_0^1 (-16x^5 + 24x^3 - 6x)\,dx$$

The rest is easy enough, giving $I = \frac{1}{3}(-2 + j)$ which is, of course, the same result as before. Note that all results in Frames 25–28 can be obtained very easily by integrating the function of z with respect to z.

For example, the integral $\int_c f(z)\,dz$ where $f(z) = (z - j)^2$ and c is the straight line joining A $(z = 0)$ to B $(z = 1 + j2)$ can be evaluated as

$$\int_c f(z)\,dz = \int_{z=0}^{1+j2} (z - j)^2\,dz$$
$$= \left[\frac{(z - j)^3}{3}\right]_0^{1+j2}$$
$$= \left(\frac{(1 + j2 - j)^3}{3} - \frac{(-j)^3}{3}\right)$$
$$= \frac{1}{3}(-2 + j)$$

Now on to the next frame

Cauchy's theorem

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We have already seen that if $w = f(z)$ where, as usual, $w = u + jv$ and $z = x + jy$, then $dz = dx + j dy$ and

$$\int f(z)\,dz = \int (u + jv)(dx + j\,dy)$$
$$= \int (u\,dx - v\,dy) + j\int (v\,dx + u\,dy)$$

If c is a closed curve as the path of integration, then

$$\oint_c f(z)\,dz = \oint_c (u\,dx - u\,dy) + j\oint_c (v\,dx + u\,dy)$$

▶

Applying Green's theorem to each of the two integrals on the right-hand side in turn, we have

(a) $\oint_c (u\,dx - v\,dy) = \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

where S is the region enclosed by the curve c.

Also, if $f(z)$ is regular at every point within and on c, then the Cauchy–Riemann equations give

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and therefore $-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$\therefore \quad \oint_c (u\,dx - v\,dy) = 0 \qquad (1)$

(b) Similarly, with the second integral, we have

....

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$$\boxed{\oint_c (v\,dx + u\,dy) = 0}$$

Because

$\oint_c (v\,dx + u\,dy) = \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$

Again, if $f(z)$ is regular at every point within and on c, then the Cauchy–Riemann equations give

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and therefore $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$

$\therefore \quad \oint_c (v\,dx + u\,dy) = 0 \qquad (2)$

Combining the two results (1) and (2) we have the following result.

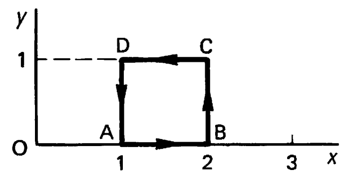
If $f(z)$ is regular at every point within and on a closed curve c, then $\oint_c f(z)\,dz = 0$

This is Cauchy's theorem. Make a note of the result; then we can see an example

31

Example 1

Verify Cauchy's theorem by evaluating the integral $\oint_c f(z)\,dz$ where $f(z) = z^2$ around the square formed by joining the points $z = 1$, $z = 2$, $z = 2 + j$, $z = 1 + j$.



$$z = x + jy$$
$$z^2 = x^2 - y^2 + j\,2xy$$
$$dz = dx + j\,dy$$

$$\oint_c f(z)\,dz = \oint_c z^2\,dz = \oint_c \{x^2 - y^2 + j\,2xy\}\{dx + j\,dy\}$$
$$= \oint_c \{(x^2 - y^2)\,dx - 2xy\,dy\} + j\oint_c \{2xy\,dx + (x^2 - y^2)\,dy\}$$

We now take each of the sides in turn.

(a) AB: $y = 0 \quad \therefore\ dy = 0$

$$\therefore \quad \int_{AB} f(z)\,dz = \int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(b) BC: $x = 2 \quad \therefore\ dx = 0$

$$\therefore \quad \int_{BC} f(z)\,dz = \int_0^1 (-4y\,dy) + j\int_0^1 (4 - y^2)\,dy$$
$$= \left[-2y^2\right]_0^1 + j\left[4y - \frac{y^3}{3}\right]_0^1$$
$$= -2 + j\left(4 - \frac{1}{3}\right) = -2 + j\frac{11}{3}$$

Continuing in the same way, the results for the remaining two sides are

.... and ....

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$$\boxed{\text{CD: } -\frac{4}{3} - j3; \quad \text{DA: } 1 - j\frac{2}{3}}$$

Because

(c) CD: $y = 1 \quad \therefore\ dy = 0$

$$\therefore \quad \int_{CD} f(z)\,dz = \int_2^1 (x^2 - 1)\,dx + j\int_2^1 2x\,dx$$
$$= \left[\frac{x^3}{3} - x\right]_2^1 + j\left[x^2\right]_2^1 = -\tfrac{4}{3} - j3$$

▶

(d) DA: $x = 1 \quad \therefore \; dx = 0$

$$\therefore \int_{DA} f(z)\,dz = \int_1^0 (-2y\,dy) + j\int_1^0 (1 - y^2)\,dy$$

$$= \left[-y^2\right]_1^0 + j\left[y - \frac{y^3}{3}\right]_1^0 = 1 - j\tfrac{2}{3}$$

So, collecting the four results, $\oint_c f(z)\,dz = \ldots\ldots\ldots\ldots$

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$$\boxed{\oint_c f(z)\,dz = 0}$$

Because

$$\oint_c f(z)\,dz = \frac{7}{3} + \left(-2 + j\frac{11}{3}\right) + \left(-\frac{4}{3} - j3\right) + \left(1 - j\frac{2}{3}\right) = 0$$

Example 2

A region in the z-plane has a boundary c consisting of

(a) OA joining $z = 0$ to $z = 2$

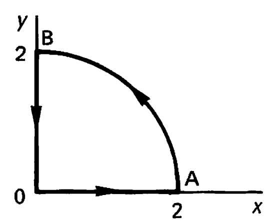
(b) AB a quadrant of the circle $|z| = 2$ from $z = 2$ to $z = j2$

(c) BO joining $z = j2$ to $z = 0$.

Verify Cauchy's theorem by evaluating the integral $\int_c (z^2 + 1)\,dz$

(1) along the arc from A to B

(2) along BO and OA.



$$f(z) = z^2 + 1 = (x + jy)^2 + 1$$

$$= (x^2 - y^2 + 1) + j2xy$$

$$z = x + jy \quad \therefore \; dz = dx + j\,dy$$

So the general expression for $\int f(z)\,dz = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{aligned}&\int\{(x^2-y^2+1)+j\,2xy\}\{dx+j\,dy\}\\ &=\int\{(x^2-y^2+1)\,dx-2xy\,dy\}+j\int\{2xy\,dx+(x^2-y^2+1)\,dy\}\end{aligned}}$$

(1) Arc AB: $x^2+y^2=4 \quad \therefore\ y^2=4-x^2 \quad \therefore\ y=\sqrt{4-x^2}$

$$dy=\frac{1}{2}(4-x^2)^{-1/2}(-2x)\,dx \qquad \therefore\ dy=\frac{-x}{\sqrt{4-x^2}}dx$$

$$\therefore\ \int_{AB} f(z)\,dz$$

$$=\int_2^0\left\{(x^2-4+x^2+1)\,dx-2x\sqrt{4-x^2}\left(\frac{(-x)}{\sqrt{4-x^2}}\right)dx\right\}$$

$$+j\int_2^0\left\{2x\sqrt{4-x^2}\,dx+(x^2-4+x^2+1)\left(\frac{(1-x)}{\sqrt{4-x^2}}\right)dx\right\}$$

$$=\int_2^0(4x^2-3)\,dx+j\int_2^0\frac{11x-4x^3}{\sqrt{4-x^2}}\,dx=-\frac{14}{3}+jI_1$$

Now we must attend to $I_1=\displaystyle\int_2^0\frac{11x-4x^3}{\sqrt{4-x^2}}\,dx$.

Substituting $x=2\sin\theta$ and $dx=2\cos\theta\,d\theta$ with appropriate limits we have

. . . .

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$$\boxed{I_1=-\frac{2}{3}}$$

Because

$$I_1=\int_{\pi/2}^0\left(\frac{22\sin\theta-32\sin^3\theta}{2\cos\theta}\right)2\cos\theta\,d\theta$$

$$=\int_0^{\pi/2}(32\sin^3\theta-22\sin\theta)\,d\theta$$

$$=32\frac{2}{(3)(1)}+\Big[22\cos\theta\Big]_0^{\pi/2}=\frac{64}{3}-22=-\frac{2}{3}$$

$$\therefore\ \int_{AB} f(z)\,dz=-4\frac{2}{3}-j\frac{2}{3}=-\frac{2}{3}(7+j)$$

(2) Along BO and OA. Complete this section on your own in the same way.

$$\int_{BO} f(z)\,dz=\ldots\ldots\ldots\ldots; \quad \int_{OA} f(z)\,dz=\ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{\text{BO}} f(z)\,\mathrm{d}z = j\frac{2}{3};\quad \int_{\text{OA}} f(z)\,\mathrm{d}z = 4\frac{2}{3}}$$

Because we have

BO: $x = 0 \quad \therefore \ \mathrm{d}x = 0$

$$\therefore \quad \int_{\text{BO}} f(z)\,\mathrm{d}z = j\int_2^1 (1 - y^2)\,\mathrm{d}y = j\left[y - \frac{y^3}{3}\right]_2^0 = j\frac{2}{3}$$

OA: $y = 0 \quad \therefore \ \mathrm{d}y = 0$

$$\therefore \quad \int_{\text{OA}} f(z)\,\mathrm{d}z = \int_0^2 (x^2 + 1)\,\mathrm{d}x = \left[\frac{x^3}{3} + x\right]_0^2 = 4\frac{2}{3}$$

Collecting the results together, therefore

$$\int_{\text{AB}} f(z)\,\mathrm{d}z = -\frac{14}{3} - j\frac{2}{3}$$

$$\int_{\text{BO+OA}} f(z)\,\mathrm{d}z = j\frac{2}{3} + 4\frac{2}{3} = \frac{14}{3} + j\frac{2}{3}$$

$$\therefore \quad \oint_{\text{c}} f(z)\,\mathrm{d}z = \int_{\text{AB}} f(z)\,\mathrm{d}z + \int_{\text{BO+OA}} f(z)\,\mathrm{d}z = 0$$

which, once again, verifies Cauchy's theorem.

Just by way of revision, Cauchy's theorem actually states that

....

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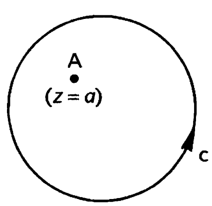
If $f(z)$ is *regular* at every point within and on a closed curve c, then $\oint_{\text{c}} f(z)\,\mathrm{d}z = 0$

In our examples so far, $f(z)$ has been regular and no problems have arisen. Let us now consider a case where one or more singularities occur within the region enclosed by the curve c.

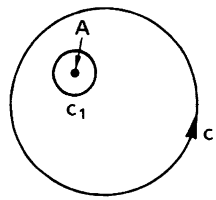
Deformation of contours at singularities



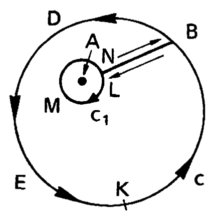
If c is the boundary curve (or *contour*) of a region and $f(z)$ is regular for all points within and on the contour, then the evaluation of $\oint_{\text{c}} f(z)\,\mathrm{d}z$ around the contour is straightforward.

However, if $f(z) = \dfrac{1}{z - a}$, where a is a complex constant, and point A corresponds to $z = a$, then at A, $f(z)$ ceases to be regular and a singularity occurs at that point.

▶



We can isolate A in a very small region within a contour c_1 and then $f(z)$ will be regular at all points within the region c and outside c_1. But the original region is now no longer simply connected (it now has a 'hole' in it) and this was one of our initial conditions.



However, all is not lost! We select a suitable point B on the contour c and join it to the inner contour c_1. If we now consider the integration $\int f(z)\,dz$ starting from a point K and proceeding anticlockwise, the path of integration can be taken as K B L M N B D E K.

Therefore

$$\int f(z)\,dz = I = I_{KB} + I_{BL} + I_{LMN} + I_{NB} + I_{BDEK} = \ldots\ldots\ldots\ldots$$

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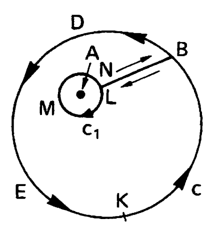
$\boxed{0}$

The function $f(z)$ is now regular at all points within and on the deformed contour. Remember that the inner contour c_1 can be made as small as we wish.

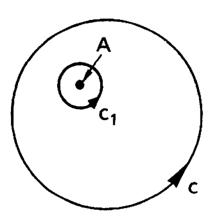


Note that $I_{NB} = -I_{BL}$, being in opposite directions, and these therefore cancel out.

The previous result then becomes

$I_{KB} + I_{LMN} + I_{BDEK} = 0$ i.e. $I_{KB} + I_{BDEK} + I_{LMN} = 0$

But $I_{KB} + I_{BDEK} = \oint_c f(z)\,dz$ and $I_{LMN} = \ointclockwise_{c_1} f(z)\,dz$



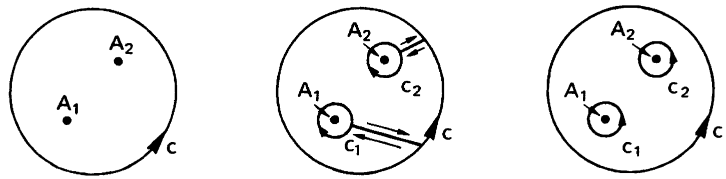
$$\therefore \quad \oint_c f(z)\,dz + \ointclockwise_{c_1} f(z)\,dz = 0$$

$$\therefore \quad \oint_c f(z)\,dz - \oint_{c_1} f(z)\,dz = 0$$

$$\therefore \quad \oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz$$

▶

The process can, of course, be extended to cases with more than one such singularity.



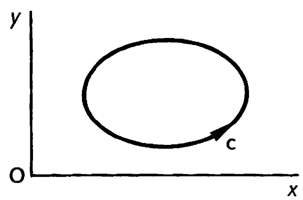
The corresponding result then becomes

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz \ldots \text{etc.}$$

Now let us apply these ideas to an example.

Example 1

Consider the integral $\oint_C f(z)\,dz$ where $f(z) = \dfrac{1}{z}$, evaluated round a closed contour in the z-plane.



We first check the function $f(z) = \dfrac{1}{z}$ for singularities and find at once that

....

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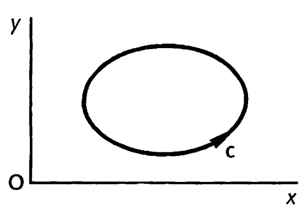
$\boxed{\text{At } z = 0,\ f(z) = \dfrac{1}{z} \text{ ceases to be regular and a singularity occurs at that point}}$

The actual position of the closed contour is not specified in the problem, so there are two possibilities: either the contour does enclose the origin, or it does not.

Let us consider them in turn.

(a) The contour does not enclose the origin.



No difficulty arises here and by Cauchy's theorem

....

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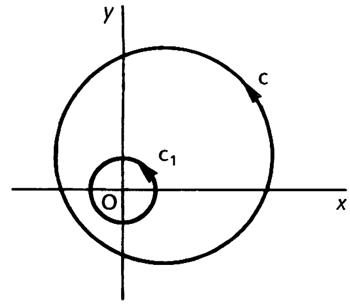
$$\boxed{\oint_c f(z)\,dz = 0}$$

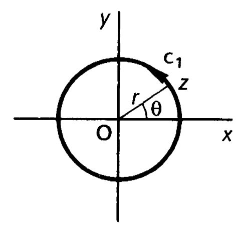
(b) If the contour *does* enclose the origin, the singularity must be taken into account. Then

$$\oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz = \oint_{c_1} \frac{1}{z}dz$$



and we attend to evaluating $\oint_{c_1} \frac{1}{z}dz$ where c_1 is a small circle of radius r entirely within the region bounded by c.

If we take an enlarged view of the small circle c_1, we have $z = x + jy$ which can be expressed in polar form .... and in exponential form ....



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$$\boxed{\begin{aligned} z &= r\,(\cos\theta + j\sin\theta) \\ z &= re^{j\theta} \end{aligned}}$$

Using $z = re^{j\theta}$ then $dz = jre^{j\theta}d\theta$ and $\oint_{c_1} \frac{1}{z}dz = \ldots\ldots\ldots\ldots$

Complete it

42

$$\boxed{j2\pi}$$

Because

$$\oint_{c_1} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{re^{j\theta}}\{jre^{j\theta}\}d\theta = \int_0^{2\pi} j\,d\theta = j2\pi$$

$$\therefore \oint_c \frac{1}{z}dz = \oint_{c_1} \frac{1}{z}dz = j2\pi$$

So we have:

(a) $\oint_c \frac{1}{z}dz = 0$ if the contour c does not enclose the origin

(b) $\oint_c \frac{1}{z}dz = j2\pi$ if the contour c does enclose the origin.

These two constitute an important result, so note them well

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Example 2

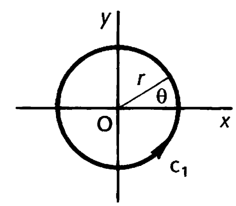
Consider the integral $\oint_c f(z)\,dz$ where $f(z) = \dfrac{1}{z^n}$ $(n = 2, 3, 4, \ldots)$.

Again, a singularity clearly occurs at $z = 0$ and again also we have two possible cases.

(a) If the contour c does not enclose the origin, then by Cauchy's theorem $\oint_c f(z)\,dz = 0$.

(b) If the contour c does enclose the origin, then we proceed very much as before.

Using $z = re^{j\theta}$, $dz = jre^{j\theta}d\theta$ and $z^n = r^n e^{jn\theta}$



Then
$$\oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz$$
$$= \int_0^{2\pi} \frac{1}{r^n e^{jn\theta}}\{jre^{j\theta}\}\,d\theta$$
$$= \frac{j}{r^{n-1}}\int_0^{2\pi} e^{-j(n-1)\theta}d\theta$$
$$= \frac{-1}{(n-1)r^{n-1}}\left[e^{-j(n-1)\theta}\right]_0^{2\pi}$$
$$= \ldots\ldots\ldots\ldots$$

Finish it off

44

$$\boxed{0}$$

Because
$$\oint_c \frac{1}{z^n}dz = \frac{-1}{(n-1)r^{n-1}}\{e^{-j(n-1)2\pi} - 1\}$$
$$= \frac{-1}{(n-1)r^{n-1}}\{\cos(n-1)2\pi - j\sin(n-1)2\pi - 1\}$$
$$= 0 \quad \text{since} \left.\begin{array}{r}\cos(n-1)2\pi = 1\\ \sin(n-1)2\pi = 0\end{array}\right\} n = 2, 3, 4, \ldots$$

So $\oint_c \dfrac{1}{z^n}dz = 0$ for all positive integer values of n other than $n = 1$, where c is any closed contour.

The particular case when $n = 1$ we have seen in Example 1.

Now we can easily cope with this next example.

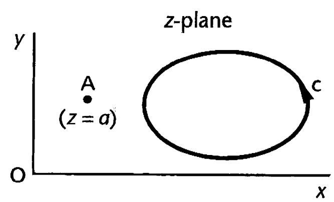
▶

Example 3

Consider $\oint_c f(z)\,\mathrm{d}z$ where $f(z) = \dfrac{1}{(z-a)^n}$ for $n = 1, 2, 3, \ldots$

This is a simple extension of the previous piece of work. Here we see that a singularity occurs at $z = a$ and yet again we have two cases to consider.

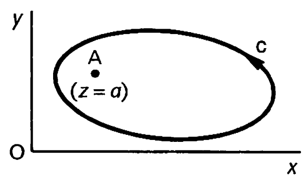
(a)



If the contour c does not enclose $z = a$, then by Cauchy's theorem

$$\oint_c f(z)\,\mathrm{d}z = 0$$

(b)



If c encloses A ($z = a$) we consider separately the cases when

(1) $n = 1$ and (2) $n > 1$.

(1) If $n = 1$, $\displaystyle\oint_c f(z)\,\mathrm{d}z = \oint_c \frac{1}{z-a}\,\mathrm{d}z$

Putting $z - a = w$ $\therefore$ $\mathrm{d}z = \mathrm{d}w$ $\therefore$ $\displaystyle\oint_c \frac{1}{z-a}\,\mathrm{d}z = \oint_c \frac{1}{w}\,\mathrm{d}w$

and this we have already established has a value ....

45

$$\boxed{j2\pi}$$

(2) If $n > 1$, $\displaystyle\oint_c f(z)\,\mathrm{d}z = \oint_c \frac{1}{(z-a)^n}\,\mathrm{d}z = \oint_c \frac{1}{w^n}\,\mathrm{d}w = 0$ for $n \neq 1$.

So collecting our results together, we have the following.

For $\oint_c f(z)\,\mathrm{d}z$, where $f(z) = \dfrac{1}{(z-a)^n}$, $n = 1, 2, 3, \ldots$ and c is a closed contour

$$\begin{aligned} \oint_c \frac{1}{(z-a)^n}\,\mathrm{d}z &= 0 && n \neq 1 \\ &= 0 && n = 1 \text{ and c does not enclose } z = a \\ &= j2\pi && n = 1 \text{ and c does enclose } z = a. \end{aligned}$$

You will notice that this is a more general result and includes the results obtained from Examples 1 and 2. *Make a note of it, therefore: it is quite important.*

Then on to Example 4

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Example 4

Finally, we can go one stage further and consider the contour integral of functions such as $f(z) = \dfrac{z - j - 4}{(z + j)(z - 2)}$.

First we express $f(z)$ in partial fractions

$$\frac{z - j - 4}{(z + j)(z - 2)} = \frac{A}{z + j} + \frac{B}{z - 2}$$

One quick way of finding A and B is by the 'cover up' method.

(a) *To find A*, temporarily cover up the denominator $(z + j)$ in the partial fraction $\dfrac{A}{[z + j]}$ and in the function $\dfrac{z - j - 4}{[z + j](z - 2)}$ and substitute $z + j = 0$, i.e. $z = -j$ in the remainder of the function.

$$A = \frac{-j - j - 4}{-j - 2} = \frac{4 + j2}{2 + j} = 2 \;\therefore\; A = 2$$

(b) *To find B*, cover up the denominator $(z - 2)$ in the partial fraction $\dfrac{B}{[z - 2]}$ and in the function $\dfrac{z - j - 4}{(z + j)[z - 2]}$ and substitute $z - 2 = 0$, i.e. $z = 2$ in the remainder of the function.

$B = \ldots\ldots\ldots\ldots$

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$$\boxed{B = -1}$$

Because

$$\begin{aligned} B &= \frac{2 - j - 4}{2 + j} \\ &= \frac{-2 - j}{2 + j} \\ &= -1 \end{aligned}$$

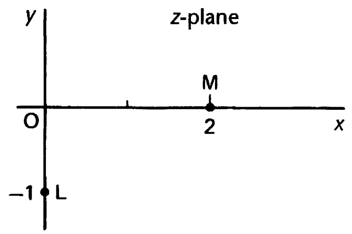
Therefore the function $f(z)$ becomes

$$f(z) = \frac{z - j - 4}{(z + j)(z - 2)} \equiv \frac{2}{z + j} - \frac{1}{z - 2}$$

Now we can see that there are singularities at $\ldots\ldots\ldots\ldots$

$$\boxed{z = -j \quad \text{and} \quad z = 2}$$

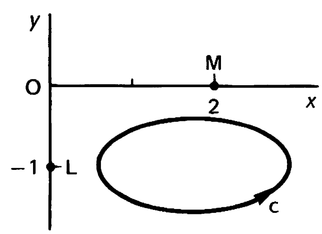
Denote the singularities by L and M.



$$\therefore \oint_c \frac{z - j - 4}{(z+j)(z-2)} \mathrm{d}z = \oint_c \left\{ \frac{2}{z+j} - \frac{1}{z-2} \right\} \mathrm{d}z$$
$$= \oint_c \left\{ 2\left(\frac{1}{z+j}\right) - \frac{1}{z-2} \right\} \mathrm{d}z$$

So we now have *four* cases to consider, depending on whether L, M, neither, or both, are enclosed within the contour c.

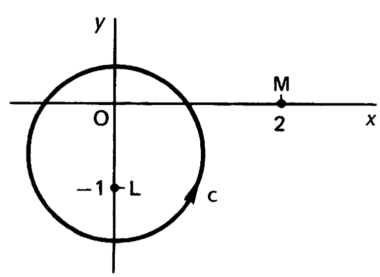
(a) *Neither L nor M enclosed*



Then, once again, by Cauchy's theorem

$$\oint_c f(z)\,\mathrm{d}z = 0$$

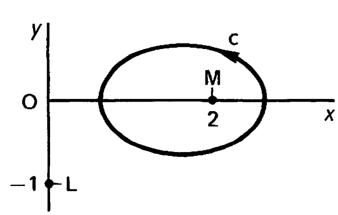
(b) *L enclosed but not M*



Then, in this case

$$\oint_c f(z)\,\mathrm{d}z = 2(j2\pi) - 0 = j4\pi$$

(c) *M enclosed but not L*

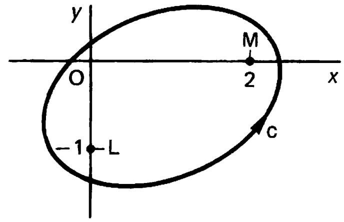


Here

$$\oint_c f(z)\,\mathrm{d}z = 0 - (j2\pi) = -j2\pi$$

▶

(d) *Both L and M enclosed*



In this case

$$\oint_c f(z)\,dz = \ldots\ldots\ldots\ldots$$

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$$\boxed{j2\pi}$$

Because, when both L and M are enclosed

$$\oint_c f(z)\,dz = \oint_c \left\{2\left(\frac{1}{z+j}\right) - \frac{1}{z-2}\right\}dz$$
$$= 2(j2\pi) - j2\pi$$
$$= j2\pi$$

The key is provided by the results we established earlier.

$$\oint_c \frac{1}{(z-a)^n}\,dz = \ldots\ldots\ldots\ldots \text{ if } \ldots\ldots\ldots\ldots$$
$$= \ldots\ldots\ldots\ldots \text{ if } \ldots\ldots\ldots\ldots$$
$$= \ldots\ldots\ldots\ldots \text{ if } \ldots\ldots\ldots\ldots$$

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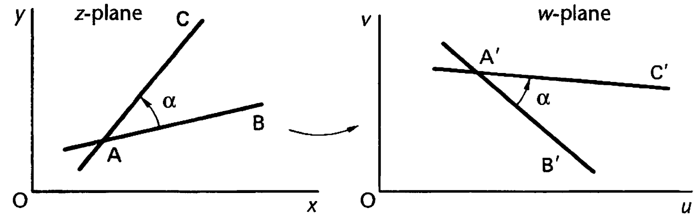
$$\begin{aligned} \oint_c \frac{1}{(z-a)^n}\,dz &= 0 && \text{if } n \neq 1 \\ &= 0 && \text{if } n = 1 \text{ and c does not enclose } z = a \\ &= j2\pi && \text{if } n = 1 \text{ and c does enclose } z = a. \end{aligned}$$

Now for something somewhat different.

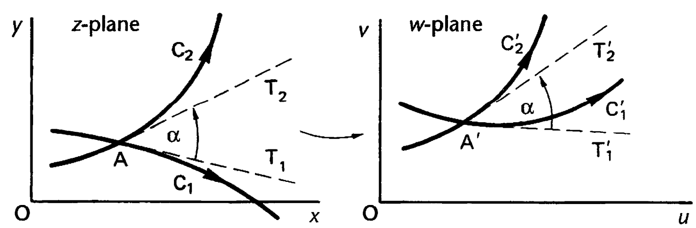
▶

Conformal transformation (conformal mapping)

A mapping from the z-plane onto the w-plane is said to be *conformal* if the angles between lines in the z-plane are preserved both in magnitude and in sense of rotation when transformed onto the corresponding lines in the w-plane.



The angle between two intersecting curves in the z-plane is defined by the angle α $(0 \leq \alpha \leq \pi)$ between their two tangents at the point of intersection, and this is preserved.



The essential characteristic of a conformal mapping is that

. . . .

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angles are preserved both in magnitude and in sense of rotation

Conditions for conformal transformation

The conditions necessary in order that a transformation shall be conformal are as follows.

1. The transformation function $w = f(z)$ must be a regular function of z. That is, it must be defined and single-valued, have a continuous derivative at every point in the region and satisfy the Cauchy–Riemann equations.
2. The derivative $\dfrac{dw}{dz}$ must not be zero, i.e. $f'(z) \neq 0$ at a point of intersection.

Critical points

A point at which $f'(z) = 0$ is called a *critical point* and, at such a point, the transformation is not conformal.

So, if $w = f(z)$ is a regular function, then, except for points at which $f'(z) = 0$, the transformation function will preserve both the magnitude of the angle and its sense of rotation.

Now for a short exercise by way of practice.

Exercise

Determine critical points (if any) which occur in the following transformations $w = f(z)$.

1 $f(z) = (z-1)^2$
2 $f(z) = e^z$
3 $f(z) = \dfrac{1}{z^2}$
4 $f(z) = z + \dfrac{1}{z}$
5 $f(z) = (2z+3)^3$
6 $f(z) = z^3 + 6z + 9$
7 $f(z) = \dfrac{z-j}{z+j}$
8 $f(z) = (z+3)(z-j)$.

Finish the whole set before checking with the results in the next frame.

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1	$z = 1$	**5**	$z = -\frac{3}{2}$
2	none	**6**	$z = \pm j\sqrt{2}$
3	none	**7**	none
4	$z = \pm 1$	**8**	$z = \frac{1}{2}(j-3)$

All that is required is to differentiate each function and to find for which values of z, $f'(z) = 0$.

Now one or two simple examples on conformal mapping.

Example 1

Linear transformation $w = az + b$, $a \neq 0$, a and b complex.

(1) Cauchy–Riemann conditions satisfied.

(2) $f'(z) = a$ i.e. not zero $\therefore$ no critical points.

Therefore, the transformation $w = az + b$ provides conformal mapping throughout the entire z-plane.

Example 2

Non-linear transformation $w = z^2$.

First check for singularities and critical points. These, if any, occur at

....

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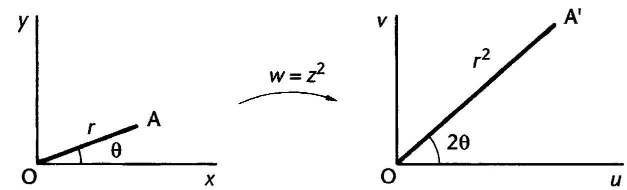
$$\boxed{\text{no singularities; critical point at } z = 0}$$

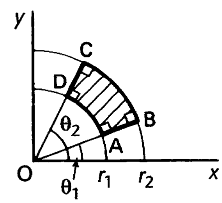
Because

$$f'(z) = 2z \quad \therefore \; f'(z) = 0 \text{ at } z = 0.$$

Therefore, the transformation is not conformal at the origin.

If we choose to express z in exponential form $z = x + jy = re^{j\theta}$, then $w = z^2 = r^2e^{j2\theta}$, i.e. r is squared and the angle doubled.





So ABCD, a section of an annulus of inner and outer radii r_1 and r_2 respectively, will be mapped onto

. . . .

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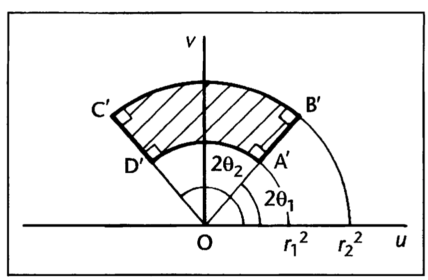

The angles at the origin are doubled, but notice that the right angles at A, B, C, D are preserved at A′, B′, C′, D′, i.e. the transformation there is conformal.

Example 3

Consider the mapping of the circle $|z| = 1$ under the transformation $w = z + \dfrac{4}{z}$ onto the w-plane.

First, as always, check for singularities and critical points. We find

. . . .

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$$\boxed{\text{singularity at } z = 0; \text{ critical points at } z = \pm 2}$$

A singularity occurs at $z = 0$, i.e. $f'(z)$ does not exist at $z = 0$. Also $f(z) = z + \frac{4}{z} \quad \therefore \ f'(z) = 1 - \frac{4}{z^2} \quad \therefore \ f'(z) = 0$ at $z = \pm 2$.
Therefore the transformation is not conformal at $z = 0$ and at $z = \pm 2$.

In fact, if we carry out the transformation $w = z + \frac{4}{z}$ on the unit circle $|z| = 1$, we get ....

Complete it: it is good revision

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$$\boxed{\text{the ellipse } \frac{u^2}{5^2} + \frac{v^2}{3^2} = 1}$$

Because

$$w = u + jv = z + \frac{4}{z}$$

$$= x + jy + \frac{4}{x + jy}$$

$$= x + jy + \frac{4(x - jy)}{x^2 + y^2}$$

$$\therefore \ u = x + \frac{4x}{x^2 + y^2}; \qquad v = y - \frac{4y}{x^2 + y^2}$$

$$|z| = 1 \quad \therefore \ x^2 + y^2 = 1 \ \therefore \ u = x(1 + 4) = 5x; \quad v = y(1 - 4) = -3y$$

$$\therefore \ x = \frac{u}{5} \ \text{ and } \ y = -\frac{v}{3}$$

Then $x^2 + y^2 = 1$ gives $\frac{u^2}{5^2} + \frac{v^2}{3^2} = 1$

The image of the unit circle is therefore an ellipse with centre at the origin; semi major axis 5; semi minor axis 3.

Now let us move on to a new section

Schwarz–Christoffel transformation

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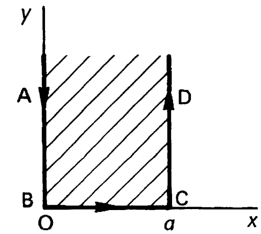
Example 1

Consider a semi-infinite strip on BC as base, the arrows at A and D indicating that the ordinate boundaries extend to infinity in the positive y-direction and that progression round the boundary is to be taken in the direction indicated.



Let us apply the transformation $w = -\cos\frac{\pi z}{a}$ to the shaded region.

Then $w = u + jv = -\cos\frac{\pi z}{a}$

$$= -\cos\frac{\pi(x+jy)}{a}$$

$$= -\left\{\cos\frac{\pi x}{a}\cos\frac{j\pi y}{a} - \sin\frac{\pi x}{a}\sin\frac{j\pi y}{a}\right\}$$

Now $\cos j\theta = \cosh\theta$ and $\sin j\theta = j\sinh\theta$.

$\therefore\ w = u + jv$

$$= -\cos\frac{\pi x}{a}\cosh\frac{\pi y}{a} + j\sin\frac{\pi x}{a}\sinh\frac{\pi y}{a}$$

$\therefore\ u = -\cos\frac{\pi x}{a}\cosh\frac{\pi y}{a};\qquad v = \sin\frac{\pi x}{a}\sinh\frac{\pi y}{a}$

So B and C map onto B′ and C′ where

B′ = ...; C′ = ...

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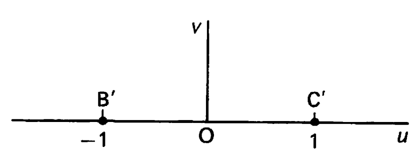
$\boxed{\text{B}':\ u = -1,\ v = 0;\quad \text{C}':\ u = 1,\ v = 0}$

Because

(1) at B, $x = 0,\ y = 0\quad \therefore\ u = -(1)(1) = -1;\ v = (0)(0) = 0$

and (2) at C, $x = a,\ y = 0\quad \therefore\ u = -(-1)(1) = 1;\ v = (0)(0) = 0$

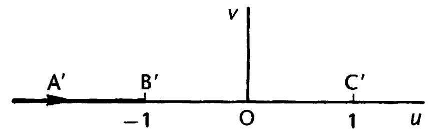
So we have



Now we map AB, BC, CD onto the w-plane giving A′B′, B′C′, C′D′.

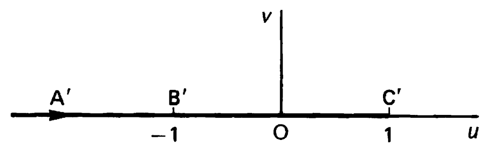
(a) AB: $x = 0 \quad \therefore$ A′B′: $u = -\cosh\dfrac{\pi y}{a}; \quad v = 0$

$\therefore$ As y decreases from ∞ to 0, u increases from $-\infty$ to -1.



(b) BC: $y = 0 \quad \therefore$ B′C′: $u = -\cos\dfrac{\pi x}{a}; \quad v = 0$

$\therefore$ As x increases from 0 to a, u increases from -1 to 1.



(c) CD: In the same way you can map CD and C′D′ in the w-plane and the mapping then becomes ....

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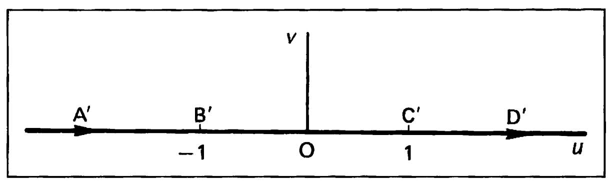
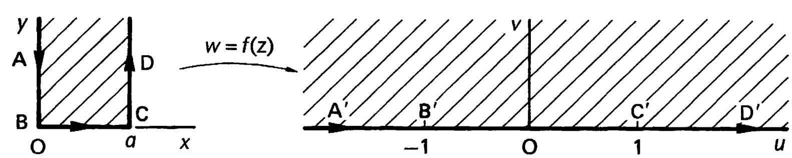


Because

CD: $x = a \quad \therefore$ C′D′: $u = \cosh\dfrac{\pi y}{a}; \quad v = 0.$

Therefore, as y increases from 0 to ∞, u increases from 1 to ∞.

Notice the direction of the arrows. These correspond to the directed travel round the boundary shown in the z-plane.



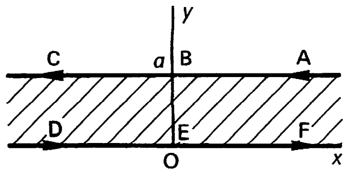
The shaded region in the z-plane is on the left-hand side of the boundary as traversed. This maps onto the left-hand side of the image on the w-plane, i.e. the entire upper half of the plane.

Note that $\dfrac{dw}{dz} = \dfrac{\pi}{a}\sin\dfrac{\pi z}{a} \quad \therefore$ at B ($z = 0$) and C ($z = a$), $\dfrac{dw}{dz} = 0$.

Therefore, the conformal property does not hold at these points. The internal angle at B and at C is $\dfrac{\pi}{2}$, while at B′ and C′ it is π.

Example 2

Consider an infinite strip in the z-plane bounded by the real axis and $z = ja$



Note the arrows. The boundary comes from $+\infty$ (A) and continues to $-\infty$ (C); then returns from $-\infty$ (D) to $+\infty$ (F).

The strip can be considered as a closed figure with the left- and right-hand vertices at infinity.

We now map the infinite strip onto the w-plane by the transformation $w = e^{\pi z/a}$.

$$\therefore \; w = u + jv = e^{\pi z/a}, \text{ from which}$$
$$u = \dots\dots\dots\dots; \; v = \dots\dots\dots\dots$$

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$$\boxed{u = e^{\pi x/a}\cos\frac{\pi y}{a}; \quad v = e^{\pi x/a}\sin\frac{\pi y}{a}}$$

Because

$$\begin{aligned} u + jv &= e^{\pi z/a} \\ &= e^{\pi(x+jy)/a} \\ &= e^{\pi x/a}e^{j\pi y/a} \\ &= e^{\pi x/a}\left(\cos\frac{\pi y}{a} + j\sin\frac{\pi y}{a}\right) \end{aligned}$$

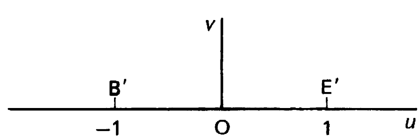
$$\therefore \; u = e^{\pi x/a}\cos\frac{\pi y}{a}; \qquad v = e^{\pi x/a}\sin\frac{\pi y}{a}$$

Now we map points B and E onto B′ and E′.

(1) B: $x = 0,\; y = a \quad \therefore$ B′: $u = -1,\; v = 0$

(2) E: $x = 0,\; y = 0 \quad \therefore$ E′: $u = 1,\; v = 0$

i.e.

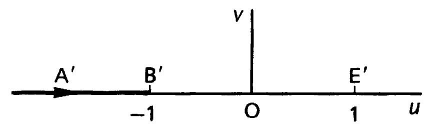


▶

Now we map the lines AB, BC, DE, EF onto the w-plane.

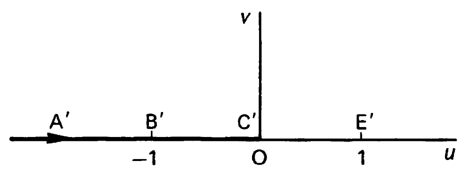
(a) AB: $y = a \quad \therefore \; u = -e^{\pi x/a}, \quad v = 0$

$\therefore$ As x decreases from $+\infty$ to 0, u increases from $-\infty$ to -1.



(b) BC: $y = a \quad \therefore \; u = -e^{\pi x/a}, \quad v = 0$ (as for AB)

$\therefore$ As x decreases from 0 to $-\infty$, u increases from -1 to 0.



(c) Now there is DE which maps onto

....

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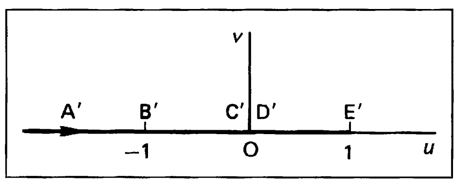


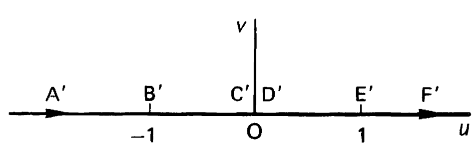
Because

(c) DE: $y = 0 \quad \therefore \; u = e^{\pi x/a}, \quad v = 0$

$\therefore$ As x increases from $-\infty$ to 0, u increases from 0 to 1.

(d) EF: $y = 0 \quad \therefore \; u = e^{\pi x/a}, \quad v = 0$ (as for DE)

$\therefore$ As x increases from 0 to $+\infty$, u increases from 1 to $+\infty$.



Notice that C and D map to the same point, namely $u = v = 0$.

Finally, what about the shaded region in the z-plane? This maps onto

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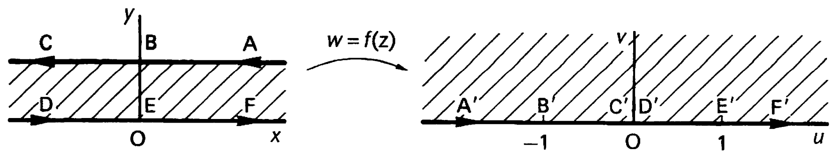
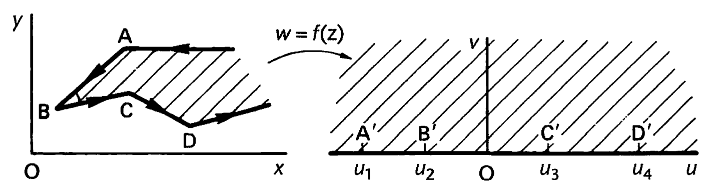
the upper half of the w-plane

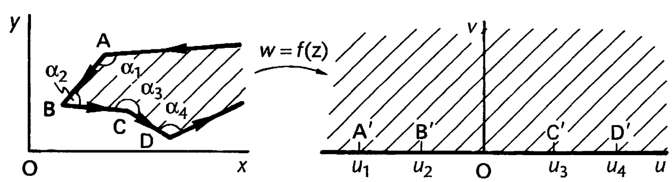
because it is on the left-hand side of the directed boundary in the z-plane.



The previous two examples have been simple cases of the application of the Schwarz–Christoffel transformation under which any polygon in the z-plane can be made to map onto the entire *upper half* of the w-plane and the boundary of the polygon onto the *real axis* of the w-plane.



The process depends, of course, on the right choice of transformation function for any particular polygon, which can be defined by its vertices and the internal angle at each vertex.



The Schwarz–Christoffel transformation function is given by

$$\frac{\mathrm{d}z}{\mathrm{d}w} = A(w-u_1)^{\alpha_1/\pi-1}(w-u_2)^{\alpha_2/\pi-1}(w-u_3)^{\alpha_3/\pi-1}\ldots$$

$$\therefore\ z = A\int(w-u_1)^{\alpha_1/\pi-1}(w-u_2)^{\alpha_2/\pi-1}\ldots(w-u_n)^{\alpha_n/\pi-1}\mathrm{d}w + B$$

where A and B are complex constants, determined by the physical properties of the polygon.

This is not as bad as it looks!

Make a careful note of it: then we will apply it

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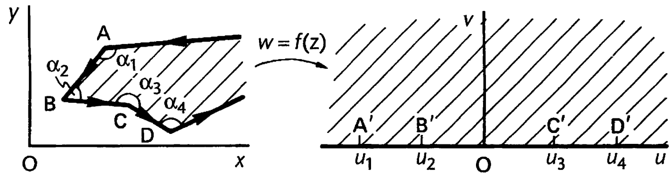
Here it is again.



$$\frac{dz}{dw} = A(w-u_1)^{\alpha_1/\pi-1}(w-u_2)^{\alpha_2/\pi-1}(w-u_3)^{\alpha_3/\pi-1}\ldots$$

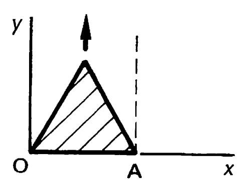
$$\therefore\ z = A\int(w-u_1)^{\alpha_1/\pi-1}(w-u_2)^{\alpha_2/\pi-1}\ldots(w-u_n)^{\alpha_n/\pi-1}dw + B$$

where A and B are complex constants.

Three other points also have to be noted.

1. Any three points u_1, u_2, u_3 on the u-axis can be selected as required.
2. It is convenient to choose one such point, u_n, at infinity, in which case the relevant factor in the integral above does not occur.
3. Infinite open polygons are regarded as limiting cases of closed polygons where one (or more) vertex is taken to infinity.

Open polygons



We have already introduced these in Examples 1 and 2 of this section.

In Example 1, the semi-infinite strip is a case of a triangle with one vertex that is

....

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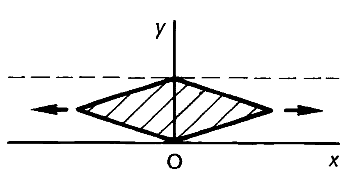
taken to infinity in the y-direction



In Example 2, the infinite strip is a case of a double triangle, or quadrilateral, with two vertices taken to infinity.

An open polygon with n sides with one vertex at infinity will have $(n-1)$ internal angles.

An open polygon with n sides with two vertices at infinity will have $(n-2)$ internal angles.

Now for an example to see how all this works.

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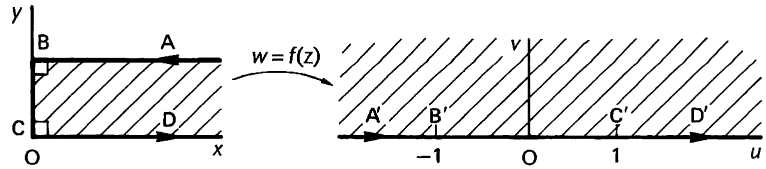
Example 3

To determine the transformation that will map the semi-infinite strip ABCD onto the w-plane so that the images of B and C occur at $u = -1$ and $u = 1$, respectively, and the shaded region maps onto the upper half of the w-plane.



In this case, B′ is $u_1 = -1$ and C′ is $u_2 = 1$.

The corresponding internal angles are:

at B $(z = ja)$, $\alpha_1 = \dfrac{\pi}{2}$ and at C $(z = 0)$, $\alpha_2 = \dfrac{\pi}{2}$.

So we have

$$\frac{dz}{dw} = A(w+1)^{(\pi/2)/\pi - 1}(w-1)^{(\pi/2)/\pi - 1} \quad \text{where } A \text{ is a complex constant}$$

$$= A(w+1)^{-1/2}(w-1)^{-1/2}$$

$$= A(w^2-1)^{-1/2}$$

$$= K(1-w^2)^{-1/2} = \frac{K}{\sqrt{1-w^2}}$$

$$\therefore \ z = \int \frac{K}{\sqrt{1-w^2}} dw = \dots\dots\dots\dots$$

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$$\boxed{z = K \arcsin w + \overline{B}}$$

$$\therefore \ \arcsin w = \frac{z - \overline{B}}{K} \qquad \therefore \ w = \sin\frac{z - \overline{B}}{K}$$

Now we have to find $\overline{B}$ and K.

(a) We require B $(z = ja)$ to map onto B′ $(w = -1)$

$$\therefore \ -1 = \sin\frac{ja - \overline{B}}{K}$$

$$\therefore \ \frac{ja - \overline{B}}{K} = -\frac{\pi}{2} \qquad \therefore \ 2ja - 2\overline{B} = -K\pi \qquad (1)$$

(b) We also require C $(z = 0)$ to map onto C′ $(w = 1)$ $\therefore \ 1 = \sin\dfrac{0 - \overline{B}}{K}$

$$\therefore \ -\frac{\overline{B}}{K} = \frac{\pi}{2} \qquad \therefore \ -2\overline{B} = K\pi \qquad (2)$$

Then, from (1) and (2), $\overline{B} = \dots\dots\dots\dots;\ K = \dots\dots\dots\dots;$

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$$\boxed{\bar{B} = \frac{ja}{2}; \quad K = -\frac{ja}{\pi}}$$

$$\therefore \; w = \sin\left\{\frac{z-(ja)/2}{-ja/\pi}\right\} = \sin\left\{jz\frac{\pi}{a}+\frac{\pi}{2}\right\} = \cos\frac{jz\pi}{a}$$

But $\cos j\theta = \cosh\theta \quad \therefore \; w = \cosh\dfrac{\pi z}{a}$

To verify that this is the required transformation, let us apply it to the figure given in the z-plane.

We will do that in the next frame

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We have

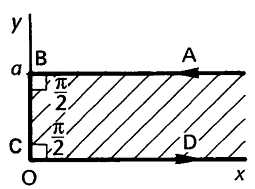
$$w = u + jv = \cosh\frac{\pi z}{a} = \cosh\frac{(x+jy)\pi}{a}$$

$$\therefore \; u + jv = \cosh\frac{x\pi}{a}\cosh\frac{jy\pi}{a} + \sinh\frac{x\pi}{a}\sinh\frac{jy\pi}{a}$$

But $\cosh j\theta = \cosh\theta$ and $\sinh j\theta = j\sin\theta$

$$\therefore \; u + jv = \cosh\frac{x\pi}{a}\cos\frac{y\pi}{a} + j\sinh\frac{x\pi}{a}\sin\frac{y\pi}{a}$$

$$\therefore \; u = \cosh\frac{x\pi}{a}\cos\frac{y\pi}{a}; \quad v = \sinh\frac{x\pi}{a}\sin\frac{y\pi}{a}$$



First map the points B and C onto B′ and C′ in the w-plane.

B′: ....; C′: ....

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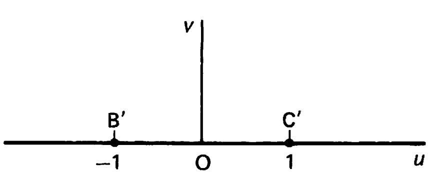
$$\boxed{\text{B}': \; u = -1, v = 0; \quad \text{C}': \; u = 1, v = 0}$$

Because

B: $x = 0, y = a \quad \therefore$ B′: $u = \cos\pi = -1, v = 0 \quad \therefore$ B′: $u = -1, v = 0$

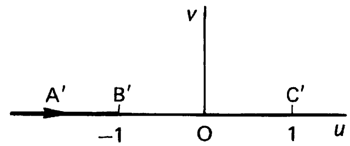
C: $x = 0, y = 0 \quad \therefore$ C′: $u = 1, v = 0 \quad \therefore$ C′: $u = 1, v = 0.$



Now we map AB, BC, CD in turn.

(a) AB: $y = a \quad \therefore\; u = -\cosh\dfrac{x\pi}{a}, \quad v = 0$

$\therefore$ As x decreases from ∞ to 0, u increases from $-\infty$ to -1.



(b) BC: } Complete the working and show the mapped region
(c) CD: }
which is ....

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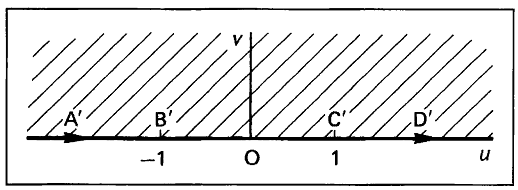

Because we have

(b) BC: $x = 0 \quad \therefore\; u = \cos\dfrac{y\pi}{a}, \quad v = 0$

$\therefore$ As y decreases from a to 0, u increases from -1 to 1.

CD: $y = 0 \quad \therefore\; u = \cosh\dfrac{x\pi}{a}, \quad v = 0$

$\therefore$ As x increases from 0 to ∞, u increases from 1 to ∞.

In each plane, the shaded region is on the left-hand side of the boundary.

We will now finish with one further example.

So move on

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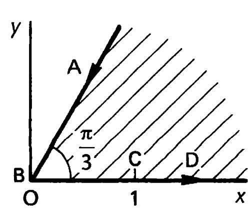
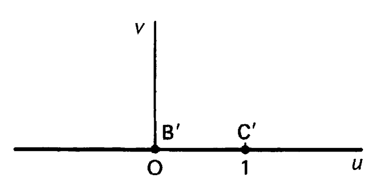
Example 4





Determine the transformation function $w = f(z)$ that maps the infinite sector in the z-plane onto the upper half of the w-plane with points B and C mapping onto B′ and C′ as shown.

The transformation function $w = f(z)$ is given by ....

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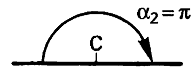
$$\frac{dz}{dw} = A(w - u_1)^{\alpha_1/\pi - 1}(w - u_2)^{\alpha_2/\pi - 1}\dots(w - u_n)^{\alpha_n/\pi - 1}$$

At B, $\alpha_1 = \frac{\pi}{3}$. At C, $\alpha_2 = \pi$.



With that reminder, you can now work through on your own, just as we did before, finally obtaining

$$w = \dots\dots\dots\dots$$

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$$\boxed{w = z^3}$$

Check with the working.

$$\frac{dz}{dw} = A(w - 0)^{(\pi/3)/\pi - 1}(w - 1)^{\pi/\pi - 1}$$
$$= Aw^{-2/3}(w - 1)^0$$
$$= Aw^{-2/3}$$
$$\therefore\ z = 3Aw^{1/3} + \overline{B}$$
$$= Kw^{1/3} + \overline{B}$$
$$\therefore\ w = \left(\frac{z - \overline{B}}{K}\right)^3$$

To find $\overline{B}$ and K

(a) At B: $z = 0$ At B′: $w = 0$ $\therefore\ 0 = \left(\frac{-\overline{B}}{K}\right)^3$ $\therefore\ \overline{B} = 0$ $\therefore\ w = \left(\frac{z}{K}\right)^3$

(b) At C: $z = 1$ At C′: $w = 1$ $\therefore\ 1 = \left(\frac{1}{K}\right)^3$ $\therefore\ K = 1$ $\therefore\ w = z^3$

$\therefore$ the transformation function is $w = z^3$

Finally, as a check – and a little more valuable practice – apply the function $w = z^3$ to the region shaded in the z-plane.

$$w = u + jv = (x + jy)^3 = x^3 + 3x^2(jy) + 3x(jy)^2 + (jy)^3$$

$$\therefore\ u = \dots\dots\dots\dots;\quad v = \dots\dots\dots\dots$$

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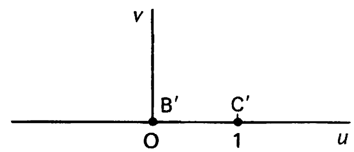
$$\boxed{u = x^3 - 3xy^2; \quad v = 3x^2y - y^3}$$

At B: $x = 0,\ y = 0 \quad \therefore\ u = 0,\ v = 0 \quad \therefore$ B′: $u = 0,\ v = 0$
At C: $x = 1,\ y = 0 \quad \therefore\ u = 1,\ v = 0 \quad \therefore$ C′: $u = 1,\ v = 0$



Now we map AB, BC, CD onto A′B′, B′C′, C′D′.

AB: $y = \sqrt{3}x \quad \therefore\ u = x^3 - 9x^3 = -8x^3, \quad v = 0$

$\therefore$ As x decreases from ∞ to 0, u increases from $-\infty$ to 0.

You can now deal with BC and CD in the same way and finally show the transformed region.

So we get ....

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Here is the remaining working.

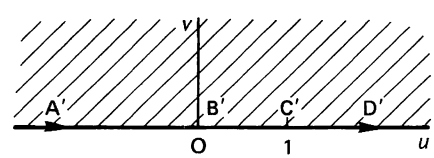
BC: $y = 0 \ \therefore\ u = x^3, \ v = 0$

$\therefore$ As x increases from 0 to 1, u increases from 0 to 1.

CD: $y = 0 \ \therefore\ u = x^3, \ v = 0$

$\therefore$ As x increases from 1 to ∞, u increases from 1 to ∞.

So we have



The shaded region is to the left of the directed boundary in the z-plane. This therefore maps onto the region to the left of the directed real axis in the w-plane, i.e. the upper half of the plane.

We have just touched on the fringe of the work on Schwarz–Christoffel transformation. The whole topic of mapping between planes has applications in fluid mechanics, heat conduction, electromagnetic theory, etc. and it is at times convenient to solve a problem relating to the z-plane by transforming to the upper half of the w-plane and later to transform back to the z-plane. The transformation function can be operated in either direction.

And that is it. The **Revision summary** follows and the **Can You?** checklist. Then on to the **Test exercise** and the **Further problems** for additional practice.

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Revision summary 21

1 *Differentiation of a complex function*

$$w = f(z) \qquad \frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0}\left\{\frac{f(z_0 + \delta z) - f(z_0)}{\delta z}\right\}$$

2 *Regular (or analytic) function*
$w = f(z)$ is *regular* at z_0 if it is defined, single-valued and has a derivative at every point at and around $z = z_0$.

3 *Singularities* or singular points – points at which $f(z)$ ceases to be regular.

4 *Cauchy–Riemann equations* test whether $w = f(z)$ has a derivative $f'(z)$ at $z = z_0$. $\quad w = u + jv = f(z)$ where $z = x + jy$.

Then $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

5 If a function of two real variables $f(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = 0$$

then $f(x, y)$ is an harmonic function. The real and imaginary parts of an analytic function are both harmonic and form a conjugate pair of functions.

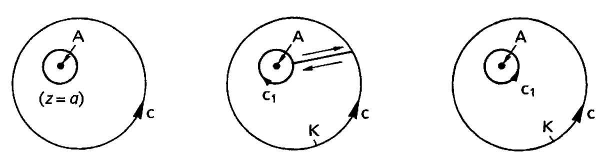
6 *Complex integration*

$$\int w\,dz = \int f(z)\,dz = \int (u\,dx - v\,dy) + j\int (v\,dx + u\,dy)$$

7 *Contour integration* – evaluation of line integrals in the z-plane.

8 *Cauchy's theorem* If $f(z)$ is regular at every point within and on closed curve c, then $\oint_c f(z)\,dz = 0$.

9 *Deformation of contours*



(a) (b) (c)

(a) Singularity at A
(b) Restored to a closed curve
(c) $\oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz.$

For $\oint_c f(z)\,dz$ where $f(z) = \dfrac{1}{(z-a)^n}$ $\quad n = 1, 2, 3, \ldots$

$$\begin{aligned} \oint_c \frac{1}{(z-a)^n}\,dz &= 0 && \text{if } n \neq 1 \\ &= 0 && \text{if } n = 1 \text{ and c does not enclose } z = a \\ &= j2\pi && \text{if } n = 1 \text{ and c does enclose } z = a. \end{aligned}$$

10 *Conformal transformation* – mapping in which angles are preserved in size and sense of rotation.

Conditions

1 $w = f(z)$ must be a regular function of z.

2 $f'(z)$, i.e. $\dfrac{dw}{dz}$, $\neq 0$ at the point of intersection.

If $f'(z) = 0$ at $z = z_0$, then z_0 is a *critical point*.

11 *Schwarz–Christoffel transformation* maps any polygon in the z-plane onto the entire *upper half* of the w-plane and the boundary of the polygon onto the *real axis* of the w-plane.

$$\frac{dz}{dw} = A(w - u_1)^{\alpha_1/\pi - 1}(w - u_2)^{\alpha_2/\pi - 1} \ldots (w - u_n)^{\alpha_n/\pi - 1}$$

1 Any three points u_1, u_2, u_3 can be selected on the u-axis.
2 One such point can be chosen at infinity.
3 Infinite open polygons are regarded as limiting cases of closed polygons.

☑ Can You?

Checklist 21

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: — Frames

- Appreciate when the derivative of a function of a complex variable exists? — 1 to 3

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Understand the notions of regular functions and singularities and be able to obtain the derivative of a regular function from first principles? — 3 to 6

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the Cauchy–Riemann equations and apply them to find the derivative of a regular function? — 7 to 12

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Understand the notion of an harmonic function and derive a conjugate function? — 13 to 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate line and contour integrals in the complex plane? — 23 to 28

  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

- Derive and apply Cauchy's theorem? 29 to 36

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply Cauchy's theorem to contours around regions that contain singularities? 37 to 49

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Define the essential characteristics of and conditions for a conformal mapping? 50 and 51

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Locate critical points of a function of a complex variable? 51 and 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the image in the w-plane of a figure in the z-plane under a conformal transformation $w = f(z)$? 52 to 56

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe and apply the Schwarz–Christoffel transformation? 57 to 75

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Text exercise 21

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1 Determine where each of the following functions fails to be regular.

(a) $w = z^3 + 4$ (d) $w = \dfrac{z-2}{(z-4)(z+1)}$

(b) $w = \dfrac{z}{z+5}$ (e) $w = \dfrac{x - jy}{x^2 + y^2}$.

(c) $w = e^{2z+4}$

2 Demonstrate that each of the following is harmonic and obtain the conjugate function.

(a) $u(x, y) = \sinh x \cos y$

(b) $u(x, y) = 4y(1 + 3x)$.

3 Verify Cauchy's theorem by evaluating $\oint_C f(z)\,dz$ where $f(z) = z^2$ round the rectangle formed by joining the points $z = 2 + j$, $z = 2 + j4$, $z = j4$, $z = j$.

4 Evaluate the integral $\oint_C f(z)dz$ where $f(z) = \dfrac{3z - 6 - j}{(z - j)(z - 3)}$ round the contour $|z| = 2$.

▶

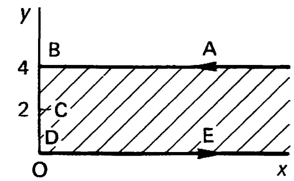
5 Determine critical points, if any, at which the following transformation functions $w = f(z)$ fail to be conformal.

(a) $w = z^4$ (d) $w = z + \dfrac{2}{z}$

(b) $w = z^3 - 3z$ (e) $w = e^{(z^2)}$

(c) $w = e^{1-z}$ (f) $w = \dfrac{z+j}{z-j}$.

6 Determine the Schwarz–Christoffel transformation function $w = f(z)$ that will map the semi-infinite strip shaded in the z-plane onto the upper half of the w-plane, so that the image of B is B′ ($w = -1$) and that of C is C′ ($w = 0$). Obtain the image of the point D.



Further problems 21

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1 Verify Cauchy's theorem for the closed path c consisting of three straight lines joining A $(1+j)$, B $(3+j3)$, C $(-1+j3)$ where $f(z) = z - 1 + j$.

2 If $z = 2 + jy$ is mapped onto the w-plane under the transformation $w = f(z) = \dfrac{1}{z}$, show that the locus of w is a circle with centre $w = 0{\cdot}25$ and radius $0{\cdot}25$.

3 Determine the image in the w-plane of the circle $|z-2| = 1$ in the z-plane under the transformation $w = (1-j)z + 3$.

4 The unit circle $|z| = 1$ in the z-plane is generated in an anticlockwise manner from the point A ($z = 1$) and is transformed onto the w-plane by $w = \dfrac{z}{z-2}$. Determine the locus of w and the direction in which it is generated.

5 Find the conjugate function of each of the following.

(a) $u(x, y) = x^2 - 2x - y^2$

(b) $u(x, y) = x^3 - 3xy^2 - x^2 + y^2 + x$

(c) $u(x, y) = 2y(x-1)$

(d) $u(x, y) = e^{x^2-y^2}\cos 2xy$.

▶

6 Evaluate $\oint_c f(z)\,dz$ where $f(z) = \dfrac{5z-2-j3}{(z-j)(z-1)}$ around the closed contour c for the two cases when

(a) c is the path $|z| = 2$

(b) c is the path $|z-1| = 1$.

7 If $f(z) = \dfrac{5z+j}{(z-j)(z+j2)}$, evaluate $\oint_c f(z)dz$ along the contours

(a) $|z-1| = 1$; (b) $|z| = \dfrac{3}{2}$; (c) $|z| = 3$.

8 If $z = x + jy$ and $w = f(z)$, show that, if $\dfrac{j(w+z)}{w-z}$ is entirely real, then $|w| = |z|$.

9 Evaluate $\oint_c f(z)\,dz$, where $f(z) = \dfrac{3z-j5}{(z+1-j2)(z-2-j)}$, around the perimeter of the rectangle formed by the lines $z = 1$, $z = j3$, $z = -2$, $z = -j$.

10 If $f(z) = \dfrac{8z^2-2}{z(z-1)(z+1)}$, evaluate $\oint_c f(z)\,dz$ along the contour c where c is the triangle joining the points $z = 2$, $z = j$, $z = -1-j$.

11 (a) For the transformation $w = z + \dfrac{1}{z}$, state (1) singularities, (2) critical points.

(b) Apply $w = z + \dfrac{1}{z}$ to map the circle $|z| = 2$ onto the w-plane.

12 Find the images in the w-plane of (a) the line $y = 0$ and (b) the line $y = x$ that result from the mapping $w = \dfrac{z-j}{z+j}$. Show that the curves intersect at the points $(\pm 1, 0)$ in the w-plane and determine the angle at which they intersect.

13 Use the transformation $w = \dfrac{j(1+z)}{1-z}$ to map the unit circle $|z| = 1$ in the z-plane onto the w-plane. Determine also the image in the w-plane of the region bounded by $|z| = 1$ and inside the circle.

14 Determine the transformation that will map the semi-infinite strip shown, onto the upper half of the w-plane, where the image of B is B′ ($w = -1$) and that of C is C′ ($w = 1$).